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AFFINE ANALOG OF THE PROPER BASE CHANGE THEOREM

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ABSTRACT

We prove a rigidity property for the étale cohomology with torsion coefficients of affine Hensel pairs.

0. Introduction

Recall that if I is an ideal in a commutative ring A then the pair (A, I) is called henselian iff the following equivalent conditions are satisfied (see [8], chap. XI, §2):

(a) $I \subset \mathrm{Rad}(A)$ (the Jacobson radical of A), and for every two relatively prime monic polynomials $\bar{g}, \bar{h} \in \bar{A}[t]$ ($\bar{A} = A/I$) and monic lifting $f \in A[t]$ of $\bar{g}\bar{h}$, there exist monic liftings $g, h \in A[t]$ s.t. $f = gh$.

(b) If B is a finite A-algebra, then $\mathrm{Idem}(B) \xrightarrow{\sim} \mathrm{Idem}(B/IB)$ (where $\mathrm{Idem}(B) =$ the Boolean algebra of idempotent elements of B).

(c) If A' is an étale A-algebra and $\sigma \in \mathrm{Hom}_{A\text{ alg}}(A', A/I)$, then $\exists_1 \bar{\sigma} \in \mathrm{Hom}_{A\text{ alg}}(A', A)$ which lifts σ.

The henselization of any pair (A, I) is (cf. ibid.) the pair (over (A, I)) $(\tilde{A}, \tilde{I}) \stackrel{\mathrm{def}}{=} (\varinjlim_{\mathcal{N}} A', \varinjlim_{\mathcal{N}} \mathrm{Ker}(\sigma))$, where $\mathcal{N}$ is the filtered category of pairs (A', σ) as in (c).*

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* $\mathcal{N}$ is filtered (in the sense of [2, I, 2.7]) since, in the category of A-algebras, finite direct limits preserve étaleness.

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THEOREM 1: *If* (A, I) *is a henselian pair,* $X = \mathrm{Spec}(A), X_0 = \mathrm{Spec}(A/I)$, *and* $i_{\mathrm{et}} : X_{0\,\mathrm{et}} \to X_{\mathrm{et}}$ *the morphism of étale sites, then for every torsion abelian sheaf* F *on* X_{et} *and* $\forall_{q \geq 0}$, *the restriction map*

$$H^q(X_{\mathrm{et}}, F) \underset{\rho_q^F}{\longrightarrow} H^q(X_{0\,\mathrm{et}}, i_{\mathrm{et}}^* F)$$

is an isomorphism.

This theorem is conjectured in ([2], XII, Remarks 6.13). We shall prove it by induction on q, and reduce by standard arguments to Lemma 1 below [which is essentially a special case of Theorem 1]. Lemma 1 is proved by induction on q using Quillen induction, the case $q = 1$ being true by Remark 1 below. Unlike the case of henselian local rings, the statement does not hold for non-torsion coefficients (take $(A, I) =$ the henselization of $(\mathbb{C}[x, y], (y^2 - x^2 - x^3)), q = 1, F = \mathbb{Z}$.) The result is used for other comparison theorems, see Fujiwara [4].

1. Constant possibly non-abelian coefficients ($q = 0, 1$)

For a set S, let $\mathbf{S}$ denote the constant sheaf defined by S on a site under consideration. Suppose (A, I) is a henselian pair.

Remark 1.1: (i) If F is a constant sheaf of sets on X_{et}, then $\Gamma(X, F) \tilde{\to} \Gamma(X_0, i^* F)$.

(ii) If G is a finite group, then $\check{H}^1_{\mathrm{et}}(X, \mathbf{G}) \tilde{\to} \check{H}^1_{\mathrm{et}}(X_0, \mathbf{G})$.

Proof: (i) follows from condition (b) for $B = A$ which means that any closed and open subset $U \subset X_0$ extends uniquely to a closed and open subset of X.

(ii) is the conclusion for isomorphism classes of

LEMMA 1.1: *The functor*

$$(\mathbf{G}\text{-torsors on } \mathrm{Spec}(A)_{\mathrm{et}}) \underset{T}{\longrightarrow} (\mathbf{G}\text{-torsors on } \mathrm{Spec}(A/I)_{\mathrm{et}})$$

is an equivalence of categories.

Proof: *T is fully faithful:* If α, β are $\mathbf{G}$-torsors on $\mathrm{Spec}(A)_{\mathrm{et}}$, then $\mathbf{Isom}(\alpha, \beta)$ is represented by a finite étale morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$, and we apply (c). [One can also use (b) for the idempotents defining the graph of an isomorphism.]

T is essentially surjective: This is shown using the methods of [3].*

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* A similar argument shows that (finite étale X schemes) $\to$ (finite étale X_0 schemes) is an equivalence of categories, which answers a question of (EGA IV, 18.5.16) in the affine case.

Definition: If P is a finitely generated projective module over a (commutative) ring K, then by an n-rigidification ($n \in \mathbf{N}$) of P we mean a pair (p, ζ), p a projector on $K^n, \zeta\colon P \tilde{\to} \mathrm{Im}(p)$.

If Z is a scheme and F is a $\mathbf{G}$-torsor on Z_{et}, then F is the sheaf of sections of a finite étale Z scheme Z'. If Z is affine, then $\mathcal{O}(Z')$ has an n-rigidification as an $\mathcal{O}(Z)$-module for all $n \gg 0$. Consider the functor $F : (\mathrm{Sch}/Z) \to (\mathrm{sets})$, associating to every Z-scheme W the set of isomorphism classes of n-rigidified $\mathbf{G}$-torsors on W_{et}.

LEMMA α (compare [3, lemma in §III2]): (i) *F is represented by an affine Z scheme Φ of finite presentation.*
(ii) *$\Phi \to Z$ is smooth.* (In fact, $\Phi \to$(scheme of projectors on $\mathcal{O}^n$) is smooth.)

Proof: For (i), one considers relative representability of $F \to$ (projectors on $\mathcal{O}^n$). (If W is a Z scheme and p is a projector on $\mathcal{O}_W^n$, then the problem of endowing $\mathrm{Im}(p)$ with an algebra structure and G action is represented by an affine f.p. morphism $W' \to W$. The étaleness condition on the resulting $\mathcal{O}_{W'}$-algebra is represented by an affine open immersion $\Omega \hookrightarrow W'$ (defined locally by inverting a discriminant); and on Ω the condition that the G action gives a torsor is closed and open.)

For (ii), it is enough (by the definition [6], 17.3) to check that F is formally smooth, i.e., that $F(B) \twoheadrightarrow F(B/I)$ whenever $\mathrm{Spec}(B)$ is a Z scheme and $I \subset B$ is a nilpotent ideal.

Indeed, using [6] (18.1.2) a G-torsor on $\mathrm{Spec}(B/I)$ lifts to a G-torsor on $\mathrm{Spec}(B)$, and one checks (using only $I \subset \mathrm{Rad}(B)$) that any rigidification lifts.

To show that T is essentially surjective, we notice that any G torsor on $\mathrm{Spec}(A/I)$ comes for $n \gg 0$ from an element of $\Phi(A/I)$, and $\Phi(A) \twoheadrightarrow \Phi(A/I)$ by [3, theorem on page 568]. ∎

Remark 1.2: If B is a finite A-algebra, then (B, IB) is a henselian pair (check (b)) [8, IX, Prop. 2(i)], and hence Lemma 1.1 holds for $(\mathrm{Spec}(B), \mathrm{Spec}(B/I))$.

2. Reduction of Theorem 1 to Lemma 1

We first recall some properties of henselization. Let P be the category of pairs and $P_h \hookrightarrow_j P$ the strictly full subcategory of henselian pairs. One checks, e.g., using condition (a), that a filtered direct limit of henselian pairs is henselian [8, XI, prop. 2(ii)], so it serves as a direct limit in P_h (as well as in P).

LEMMA 2.1: *If $(A, I) = \varinjlim(A_i, I_i)$ is a filtered direct limit (in P), then the henselizations satisfy* $\varinjlim(\tilde{A}_i, \tilde{I}_i) \xrightarrow{\sim} (\tilde{A}, \tilde{I})$.

Proof: By [8, XI, def. 4, th. 2], the henselization functor sends P into P_h and it is a left adjoint to j. As any left adjoint, it preserves direct limits. (Lemma 2.1 could also be shown from the construction and results of [6]IV.)

LEMMA 2.2: (cf. [8, p. 125]) *If A is a commutative noetherian ring and $I \subset A$ an ideal, then the henselization $\tilde{A}$ of A w.r.t. I is noetherian.*

Proof: Consider a $(A', \sigma) \in \mathcal{N}$. Since A' is a formally étale A-algebra (in the sense of [6, IV, 17.1]), we have that $(\forall n \in \mathbf{N}^+)$ σ lifts uniquely to $\sigma_n \in \mathrm{Hom}_{A\ \mathrm{alg}}(A', A/I^n)$, and that the composition

$$A' \xrightarrow[\sigma_n]{} A/I^n \xrightarrow[t_n]{} A'/\ker(\sigma)^n$$

is the canonical projection. This and $\sigma_n(\ker(\sigma)^n) = 0$ give that t_n is an isomorphism. So $\varprojlim\ t_n : \hat{A} \xrightarrow{\sim} \hat{A}'$. Consider the flat composed homomorphism $A' \to \hat{A}' \xrightarrow{\sim} \hat{A}$. Taking the direct limit over $\mathcal{N}$ gives a homomorphism $\tilde{A} \xrightarrow{\varphi} \hat{A}$, which is still flat by [6, 0_{I}, (6.2.3)]. Any maximal ideal $m \subset \tilde{A}$ is contracted from a maximal ideal of $\hat{A}$ since $m \supset \mathrm{Rad}(\tilde{A}) \supset \tilde{I}$ and $A/I \xrightarrow{\sim} \tilde{A}/\tilde{I} \xrightarrow{\sim} \hat{A}/\hat{I}$. Hence φ is faithfully flat. But $\hat{A}$ is noetherian, so by [6, 0_{I}, (6.5.2)] $\tilde{A}$ is noetherian.

Proof of Theorem 1 by Induction: Let $q \geqq 0$ be given and assume that ρ^F_{q-1} is bijective for every F in the category C of torsion abelian sheaves on X_{et}.

CLAIM 1: $\forall F \in \mathrm{Ob}(C), \rho^F_q$ *is injective.*

Proof: If $q = 0$, the injectivity follows from the fact that X is the only open neighbourhood of X_0 in X, which is equivalent to $I \subset \mathrm{Rad}(A)$. We now assume $q > 0$. Embed F in an injective object G of the category C. Then we get a morphism of exact sequences

$$(*)\qquad \begin{array}{ccccccc} \cdots \to & H^{q-1}(X, G/F) & \longrightarrow & H^q(X, F) & \longrightarrow & H^q(X, G) \to \cdots \\ & \Big\downarrow \rho^{G/F}_{q-1} & & \Big\downarrow & & \Big\downarrow \\ \cdots \to & H^{q-1}(X_0, G/F) & \longrightarrow & H^q(X_0, F) & \longrightarrow & H^q(X_0, G) \to \cdots \end{array}$$

We claim that $H^q(X_{\mathrm{et}}, G) = 0$. Indeed $\forall n \in \mathbf{N}_+$, the sheaf $\mathrm{Ann}_G(n) = \mathbf{Hom_Z}(\mathbf{Z}/(n), G)$ is an injective $\mathbf{Z}/(n)$ Module, so $H^q(X, \mathrm{Ann}_G(n)) = 0$ (notice that cohomology is independent of the base Ring ([2], V, cor. 3.5)), and $G = \varinjlim \mathrm{Ann}_G(n)$ (over $\mathbf{N}_+$ ordered by divisibility) implies by ([2], VII, prop. 3.3) that $H^q(X_{\mathrm{et}}, G) \xleftarrow{\sim} \varinjlim H^q(X_{\mathrm{et}}, \mathrm{Ann}_G(n))$.

Now from (*) and the induction hypothesis one deduces the injectivity of ρ_q^F.

CLAIM 2: *ρ_q^F is bijective $\forall F \in \mathrm{Ob}(C)$.*

Proof: (i) By ([2], IX, cor. 2.7.2) F is a filtered direct limit of constructible abelian sheaves F_α. One can replace the F_α's by constructible torsion sheaves, because $\forall_\alpha$ the canonical homomorphism $F_\alpha \to F$ factorizes through F_α/nF_α for some $n > 0$. By ([2], VII, prop. 3.3) $\varinjlim_\alpha H^q(X_{\mathrm{et}}, F_\alpha) \xrightarrow{\sim} H^q(X_{\mathrm{et}}, F)$, and similarly on X_0. Thus Claim 2 is reduced to case F is constructible.

(ii)* Using (2.1) we express (A, I) $(\xrightarrow{\sim}(\tilde{A}, \tilde{I}))$ as $\varinjlim(\tilde{A}_\alpha, \tilde{I}_\alpha)$, where the A_α are the finitely generated subrings of A and $(\tilde{A}_\alpha, \tilde{I}_\alpha)$ is the henselization of $(A_\alpha, I \cap A_\alpha)$. $X = \mathrm{Spec}(A)$ is the inverse limit of the $X_\alpha := \mathrm{Spec}(\tilde{A}_\alpha)$, and by ([2], IX, cor. 2.7.4) F is isomorphic to the pull-back of a constructible torsion sheaf F_α on $X_{\alpha\,\mathrm{et}}$ for some α. We may assume that α is an initial object of the index category J. (Replace J by $\alpha \setminus J$.)

Then if $\forall \beta \in \mathrm{Ob}J$ we let F_β denote the pull-back of F_α by $X_\beta \to X_\alpha$, we have, by ([2], VII, Cor. 5.8), that

$$\varinjlim_{\beta} H^q(X_{\beta\,\mathrm{et}}, F_\beta) \xrightarrow{\sim} H^q(X, F).$$

(Similarly for X_0.) This enables one to reduce Claim 2 to the case where, in addition, A is noetherian.

(iii) If F embeds in a torsion sheaf G, then by applying the five lemma to the morphism of exact sequences

$$\begin{array}{ccccccc}
\cdots \to H^{q-1}(X, G/F) & \xrightarrow[\delta]{} & H^q(X, F) & \longrightarrow & H^q(X, G) & \longrightarrow & H^q(X, G/F) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to H^{q-1}(X_0, G/F) & \xrightarrow[\delta]{} & H^q(X_0, F) & \longrightarrow & H^q(X_0, G) & \longrightarrow & H^q(X_0, G/F)
\end{array}$$

and using Claim 1 and the induction hypothesis, we reduce Claim 2 to showing the bijectivity of ρ_q^G.

(iv) $\forall x \in \mathrm{Spec}(A)$, let x also denote the scheme $\mathrm{Spec}(k(x))$. Choose an algebraic closure $\overline{k(x)}$ of $k(x)$, and let $\bar{x} = \mathrm{Spec}(\overline{k(x)})$. We have a morphism

* Cf. the proof of ([2], XII, lemma 8.1).

$\bar{x} \underset{i_{\bar{x}}}{\to} X. i_{\bar{x}}$ factorizes through the normalization Z of $\overline{\{x\}}$ (with the integral scheme structure) in $k(\bar{x})$:

$$\bar{x} \xrightarrow[j]{} Z \xrightarrow[\pi]{} X, \qquad i_{\bar{x}} = \pi \circ j$$

(j is isomorphic to the inclusion of the generic point).

(v) Using the fact that $|X|$ is noetherian and F is constructible, we can find finitely many points $x_\alpha \in X$ ($1 \leqq \alpha \leqq k$) s.t. $F \to \Pi_\alpha i_{\alpha *} i^*_\alpha F$ is injective, where $(\forall_\alpha) i_\alpha = i_{\bar{x}_\alpha}$ ([2], IX, prop. 2.14(ii)). Then (iii) shows that Claim 2 is reduced to

CLAIM 3: *If $x \in X$ and M is a torsion abelian group and $F = i_{\bar{x}*} M$, then ρ^F_q is bijective.*

Proof of Claim 3: We recall that if Y is a normal integral scheme with generic point $\eta \underset{\phi}{\to} Y$, then the strictly local rings of Y are domains ([8], VIII, th. 3(2)), and this implies ([2], IX, lemma 2.14.1) $\phi_* \mathbf{S} \xleftarrow{\sim} \mathbf{S}$ for any set S. In particular, with the notations of (iv), we get $j_* \mathbf{M} = \mathbf{M}$ so $F = \pi_* j_* \mathbf{M} = \pi_* \mathbf{M}$. Define $Z_0 = Z \times_X X_0$ and let $\pi_0 : Z_0 \to X_0$ and $\bar{i} : Z_0 \to Z$ be the projections. Since π is an integral morphism, we have ([2], VIII, cor. 5.6) $R^q \pi_* = 0$ for $q > 0$ and π_* commutes with base changes, so $i^* \pi_* \mathbf{M} \xrightarrow{\sim} \pi_{0*} \bar{i}^* \mathbf{M}$ and

$$H^q(X_{0\,\text{et}}, i^* F) \xrightarrow{\sim} H^q_{\text{et}}(X_0, \pi_0, \mathbf{M}) \xrightarrow{\sim} H^1_{\text{et}}(Z_0, \mathbf{M}).$$

So Claim 3 is equivalent to

$$(**) \qquad H^q_{\text{et}}(Z, \mathbf{M}) \xrightarrow[\rho]{\sim} H^q_{\text{et}}(Z_0, \mathbf{M}).$$

If $q = 0$ then by (b) (extended to integral algebras by a limit argument) Z_0 is connected, so ρ is $M \underset{\text{id}}{\to} M$; therefore (**) holds. Suppose $q > 0$. Then by $j_* M = \mathbf{M}$ and $R^p j_* = 0$ for $p > 0$ (as j_* is exact), we get

$$H^q_{\text{et}}(Z, \mathbf{M}) \xrightarrow{\sim} H^q_{\text{et}}(\{\bar{x}\}, \mathbf{M}) = 0,$$

so (**) reduces to a case of

LEMMA 1: *If A is a normal domain having an algebraically closed field of fractions, then for every closed subscheme $Z \subset \operatorname{Spec}(A)$ and a torsion abelian group M we have*

$$H^q_{\text{et}}(Z, \mathbf{M}) = 0 \quad \forall q > 0.$$

(For $q = 1$ we can replace "torsion abelian group" by "locally finite group".)

3. Proof of Lemma 1

Let Y be the spectrum of $\mathrm{Idem}(Z)$, as a topological space. We have a continuous map $|Z| \underset{f}{\rightarrow} Y$, defined by $f^{-1}(U_e) = U_e \, \forall e \in \mathrm{Idem}(Z)$.

The fibers of f are the connected components of Z (this is a general fact for topological spaces on which H^0 commutes with filtered $\varinjlim$, e.g., coherent spaces (in the sense of toposes [2], VI) or compact Hausdorff spaces).

We claim that for any abelian sheaf F on Z_{et}, we have

$$(3.1) \qquad H^q(Z_{\mathrm{et}}, F) \underset{\mu}{\longrightarrow} \prod_{y \in Y} H^q(f^{-1}(y)_{\mathrm{et}}, F) \text{ is injective.}$$

Proof of 3.1: We first remark that each $f^{-1}(y)$ has a unique closed subscheme structure, and furthermore it is the $\varprojlim$ of its closed and open neighbourhoods U_i. Hence by ([2], VII, cor. 5.8)

$$H^q(f^{-1}(y)_{\mathrm{et}}, F) \xleftarrow{\sim} \varinjlim_i H^1(U_{i\,\mathrm{et}}, F).$$

Hence if $\xi \in \mathrm{Ker}(\mu)$ then $\forall_y$ ξ vanishes on some closed and open neighbourhood U_y of $f^{-1}(y)$. Finitely many such $U_i = U_{y_i}$ $(1 \leqq i \leqq n)$ suffice to cover Z. Then Z is the disjoint union of the open subschemes $U_i' = U_i - \bigcup_{j<i} U_j$, so from $\xi \,|\, U_i' = 0 \, \forall_i$ we get $\xi = 0$.

(The proof can be reformulated using the Leray spectral sequence for the morphism of sites $Z_{\mathrm{et}} \to Y$.)

LEMMA 2: *If $Z = V(I)$ of Lemma 1 is connected, then*

$$(A', I') \overset{\mathrm{def}}{=} ((1+I)^{-1}A, (1+I)^{-1}I)$$

is a henselian pair.

Proof: We check condition (a). The localization by $1 + I$ ensures $I' \subset \mathrm{Rad}(A')$. Now if $f, \bar{g}, \bar{h}$ are as in the statement of (a), then as A' is also a normal domain with an algebraically closed field of fractions, f factorizes as $\Pi_{\alpha=1}^n (t - \lambda_\alpha)$ $(\lambda_\alpha \in A')$.

So if $n > 0$ then $\bar{g}(\bar{\lambda}_1)\bar{h}(\bar{\lambda}_1) = \overline{f(\lambda_1)} = 0$, but $\bar{g}(\bar{\lambda}_1)\bar{A} + \bar{h}(\bar{\lambda}_1)\bar{A} = (1)$, so locally one of $\bar{g}(\bar{\lambda}_1), \bar{h}(\bar{\lambda}_1)$ is a unit and the other is zero. By connectedness of

Spec(A/I) this holds globally. Continuing, one obtains that for some partition $\{1,2,\ldots,N\} = \Phi_1 \amalg \Phi_2$ we have

$$\bar{g} = \prod_{\alpha\in\Phi_1} (t-\bar{\lambda}_\alpha), \quad \bar{h} = \prod_{\alpha\in\Phi_2} (t-\bar{\lambda}_\alpha).$$

Define g, h by analogous formulas without bar.

Proof of Lemma 1 for $q = 1$: (3.1) is used to reduce to the case that A/I is connected, then Lemma 2 reduces to the case that (A, I) is henselian, in which we use Remark 1(ii) and $H^1_{\mathrm{et}}(\mathrm{Spec}(A), \mathbf{M}) = 0$.

Proof of Lemma 1 for $q > 1$: We assume by induction that Lemma 1 holds for $q-1$, and we shall prove it for q.

Suppose $\xi \in H^q_{\mathrm{et}}(Z, \mathbf{M})$. Every étale morphism $V \underset{e}{\to} Z$ extends locally to a separated étale morphism $V' \underset{e'}{\to} \mathrm{Spec}(A)$. We recall

LEMMA 3: (cf. [6], IV, prop. 18.10.8). *If $V \underset{f}{\to} Z$ is a separated étale morphism with Z integral normal with generic point η, then V is isomorphic as a Z-scheme to a disjoint union of open subschemes of the normalizations of Z in finite field extensions of $k(\eta)$.*

(This follows from "Zariski's main theorem" and the fact that $\mathcal{O}_V$ is integrally closed in $\mathcal{O}_V \otimes_{\mathcal{O}_Z} k(\eta)$.)

In our case Lemma 3 implies that e' is a local homeomorphism, so e is a local homeomorphism, and we can think of V as a Zariski sheaf on Z. More precisely, the étale site of Z is equivalent (as a category with topology) to the Zariski topos of Z. From this one shows that the Zariski and étale sites of Z give equivalent toposes, so $H^q_{\mathrm{Zar}}(Z, \mathbf{M}) \xrightarrow{\sim} H^q_{\mathrm{et}}(Z, \mathbf{M})$. In particular, ξ is trivial locally for the Zariski topology on Z, so the subset

$$S = \{s \in A \mid \xi|_{Z_s} = 0\} \quad [\text{where } Z_s = Z - V(s)]$$

satisfies $A = A \cdot S$. We want to show that S is an ideal. (This will give $S = A$, so $1 \in S$, i.e., $\xi = 0$.) As S is easily seen to be closed under multiplication by elements of A, it remains to show that if $s, t \in S$ then $s + t \in S$.

We may assume $s + t \neq 0$, so replacing A by $A[1/(s+t)]$ we reduce to show that if $s + t \in A^*$ and $\xi|_{Z_s} = 0$ and $\xi|_{Z_t} = 0$, then $\xi = 0$. Notice $Z = Z_s \cup Z_t$, so we have a Mayer–Vietoris exact sequence (cf. [7], III, 2.24)

(3.2)

$$\cdots \to H^{q-1}_{\mathrm{et}}(Z_s \cap Z_t, \mathbf{M}) \underset{\delta}{\to} H^q_{\mathrm{et}}(Z, \mathbf{M}) \to H^q_{\mathrm{et}}(Z_s, \mathbf{M}) \oplus H^q_{\mathrm{et}} H^q_{\mathrm{et}}(Z_t, \mathbf{M}) \to \cdots$$

which gives $\xi \in \mathrm{Im}(\delta)$. But the term $H^{q-1}(Z_s \cap Z_t, M)$ vanishes by applying (if $s, t \neq 0$) the induction hypothesis of Lemma 1 to the ring $A[1/st]$ and the pull-back of Z to its spectrum.

Query: Does Lemma 1 hold for any affine scheme Z s.t. any monic $f \in \mathcal{O}(Z)[t]$ of deg > 0 has a root?

Connectivity properties: Let A be an absolutely integrally closed domain, Z_1 and Z_2 closed subsets of Spec(A), $Z = Z_1 \cup Z_2$. The $\mathbb{Z}/n$-torsors on Z which are trivial on both Z_1 and Z_2 are classified by the cokernel of the restriction map

$$r : H^0(Z_1, \mathbb{Z}/n) \oplus H^0(Z_2, \mathbb{Z}/n) \longrightarrow H^0(Z_1 \cap Z_2, \mathbb{Z}/n),$$

so by Lemma 1 r is surjective (for $n \neq 0$). This can fail for coefficients $\mathbb{Z}$ or $\mathbb{Q}$. In particular one gets the corollary that the intersection of two connected closed subsets of Spec(A) is connected or empty (which occurred in more special cases in Artin [1, §1]).

4. Remarks on non-abelian cases

4.1 A sheaf of groups G on a site S is called ind-finite iff $\forall n \in \mathbf{N}, G^n$ is equal to its subsheaf $(G^n)_f$ consisting of the n-tuples of local sections of G which generate locally a finite subgroup of $G(U)$.* One checks that the formation of $(G^n)_f$ commutes with applying ϕ^* for a morphism of topoii. Hence if $(P_i \to \tilde{S})$ is a conservative family of points, then G is ind-finite iff all the stalks G_{P_i} are locally finite.

If X is a scheme and $\mathcal{C} \xrightarrow[\pi]{} X_{\mathrm{et}}$ is a stack (champ in the sense of [5], ch. II), then $\mathcal{C}$ is called ind-finite ([5], ch. VII, 2.2.1) iff for every local section $\sigma \in \mathcal{C}_U$ (= the category $\pi^{-1}(U)$), the sheaf $\mathbf{Aut}(\sigma)$ on U_{et} is ind-finite.

4.2 Let (X, X_0) be an affine Hensel pair.

Remark 2: (i) ρ_0^F is bijective for any sheaf of sets F on X_{et}. (This strengthens (c).)

(ii) ρ_1^F is bijective for any ind-finite sheaf of groups F on X_{et}.

Proof: (i) Use condition (b) above and the implication (b) $\Rightarrow$ (a) in ([2], XII , prop. 6.5 (i)).

* If S is topologically generated by its quasi-compact objects, the definition of ([2], IX, 1.5) can be used.

(ii) Use Lemma 1.1 applied to finite A-algebras and the implication (b) $\Rightarrow$ (a) in ([2], XII, prop. 6.5(ii)).

Remark 2 can be strengthened to

THEOREM 1′: *If $\mathcal{C}$ is an ind-finite stack on X_{et}, then the functor from cartesian sections of $\mathcal{C}$ to cartesian sections of $i^*\mathcal{C}$ is an equivalence of categories.*

Proof: This follows from Lemma 1.1 for finite A-algebras, Remark 2(i), and the following:

PROPOSITION 1: *If $X_0 \to X$ is a closed immersion of coherent schemes, then conditions* (B) *and* (C) *of* ([5], VII, §2.2.11) *imply condition* (E) *of* (ibid.).

Proposition 1 is proved in (ibid.) in the noetherian case. It holds more generally for every morphism of coherent algebraic spaces, and can be proved by extending to gerbes the techniques of ([2], IX, XII) for sheaves of groups.

Remark 3: (cf. [2], XII, 6.5(i)). The functor T of Theorem 1′ is fully faithful for *any* stack $\mathcal{C} \to X_{\text{et}}$; this follows by applying Remark 2(i) to the sheaves $\mathbf{Hom}(\sigma, \tau)$, where σ, τ are cartesian sections of $\mathcal{C}$. Specializing to the case $\mathcal{C}$ = (the stack of F-torsors), F being *any* sheaf of groups on X_{et}, this gives that ρ_1^F is injective.

5. Proper schemes

By the proper base change theorem ([2], XII, cor. 5.5), if $\mathcal{O}$ is a henselian local ring with residue field k and Y is a proper $\mathcal{O}$-scheme, then $(Y, Y \otimes k)$ satisfies the property of Theorem 1. Combining Theorems 1 and 1′ with the proper base change theorem, we obtain:

COROLLARY 1: *If (A, I) is a henselian pair, $Y \xrightarrow[\pi]{} \text{Spec}(A)$ a proper morphism, and $Y_0 = V(I\mathcal{O}_Y)$, then (Y, Y_0) satisfies the property of Theorems 1 and 1′.*

Proof of Corollary 1: (i) For abelian cohomology: Consider the cartesian diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow[i]{\hspace{2em}} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ \text{Spec}(A/I) & \xrightarrow[j]{\hspace{2em}} & \text{Spec}(A). \end{array}$$

If F is a torsion abelian sheaf on Y_{et} and $F_0 = i^*F$, then we have a morphism between the Leray spectral sequences:

$$\begin{array}{ccc} E_2^{pq} = H^p(\mathrm{Spec}A, R^q\pi_*F) & \Rightarrow & H^{p+q}(Y_{\mathrm{et}}, F) \\ \downarrow & & \downarrow \\ 'E_2^{pq} = H^p(\mathrm{Spec}A/I, R^q\pi_{0*}F_0) & \Rightarrow & H^{p+q}(Y_{0\,\mathrm{et}}, F_0). \end{array}$$

By proper base change ([2], XII, 5.1) $j^*R^q\pi_*F \xrightarrow{\sim} R^q\pi_{0*}F_0$, and $R^q\pi_*F$ is a torsion sheaf since $R^q\pi_*$ commutes with filtered direct limits (using [2], VII, 3.3). So by Theorem 1, $E_2^{pq} \xrightarrow{\sim} {}'E_2^{pq}$, and hence the morphism of abutements is an isomorphism.

(ii) For non-abelian cohomology: We use the operations f_* and f^* on stacks ([5], ch. II, 3) relative to a morphism f of toposes. If $\mathcal{C} \xrightarrow[\pi]{} S$ is a functor and $s \in \mathrm{Ob}(S)$, $\mathcal{C}_s$ denotes the category $\pi^{-1}(s)$. If $\mathcal{C} \to Y_{\mathrm{et}}$ is an ind-finite stack then the functor $\mathcal{C}_Y \to (i^*\mathcal{C})_{Y_0}$ which is to be shown an equivalence of categories identified with the composition

$$(\pi_*\mathcal{C})_X \xrightarrow[\text{"restriction"}]{} (j^*\pi_*\mathcal{C})_{X_0} \xrightarrow[b_{X_0}]{} (\pi_{0*}i^*\mathcal{C})_{X_0}.$$

The first arrow is an equivalence of categories by Theorem 1′, and the second arrow is an equivalence since the base change morphism $b : j^*\pi_*\mathcal{C} \to \pi_{0*}i^*\mathcal{C}$ is an equivalence (by proper base change for stacks ([5], VII, 2.2.2), in which the noetherianity hypothesis can be omitted in view of Proposition 1).

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